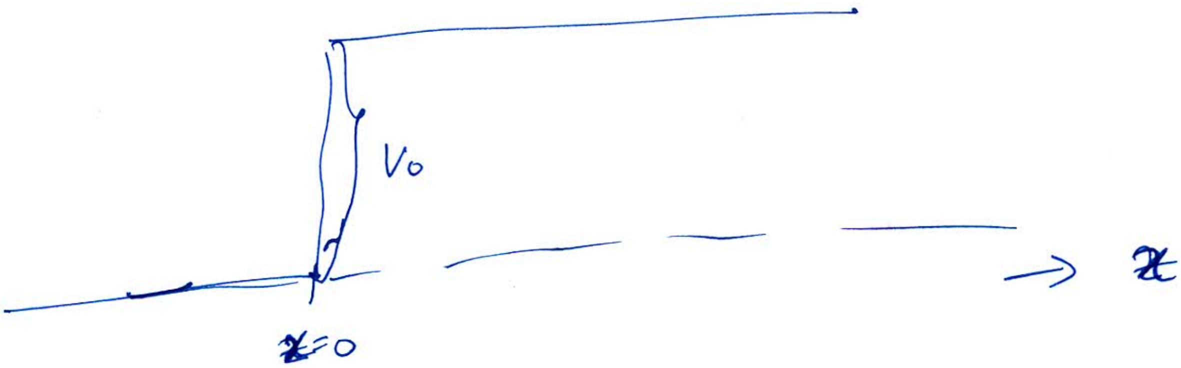
Klein ~~Klein~~ paradox

You may have first heard about the Klein paradox when you were learning relativistic quantum mechanics - when you try to solve the Dirac eqn with a barrier potential



Before we solve this problem, let me remind you of the following. In RQM, even if we start with a wave-fn. with only +ve energy components at time $t = 0$

$$\psi(\vec{x}, 0) = \begin{pmatrix} \varphi(\vec{x}) \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

the general wave-fn at any ~~time~~ later time t also has $-$ve energy components. In general, the wfn at any later time t is given by

$$\psi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m}{E}} \sum_{\pm s} \{ b(\vec{p}, s) \, u(\vec{p}, s) \, e^{-ip_\mu x^\mu} + d^*(\vec{p}, s) \, v(\vec{p}, s) \, e^{ip_\mu x^\mu} \}$$

~~At t = 0~~ Assuming that the wave-packet moves along the $\hat{x}$-direction we can easily find

$$b(p_x, s=+) = \cosh\frac{\omega}{2} \frac{1}{(2\pi)^{3/2}} \int d^3\vec{r} \, e^{-ip_x x} \phi(\vec{x})$$

$$d^*(p_x, s=-) = -\sinh\frac{\omega}{2} \frac{1}{(2\pi)^{3/2}} \int d^3\vec{r} \, e^{-ip_x x} \phi(\vec{x})$$

and $b(p_x, -) = d^*(p_x, +) = 0$

Hence

$$\frac{d^*(p_x, -)}{b(p_x, +)} = \frac{-\sinh\omega/2}{\cosh\omega/2} = \tanh\omega/2$$

$$= \frac{-p_x}{E+m} \left(\sim -\frac{p_x c}{E+mc^2} \right)$$

In the NR limit, $p_x \sim mv$ with $v \ll c$. Hence $d^*/b \sim O(v/c)$ is negligible. But -ve energy when $v \sim c$.

Also since uncertainty pple gives $\Delta\vec{p}\,\Delta\vec{x} \sim \hbar$, if mtm $\vec{p} \sim mc$, then $\Delta x \sim \frac{\hbar}{mc}$

So if the prcle is confined to distances of $O\left(\frac{\hbar}{mc}\right)$, then -ve energy components are appreciable.

One needs to keep the above pts in mind when we solve the Dirac eq^n ~~in a barrier potential~~ with a semi-infinite barrier.

$$V(\vec{x}) = V(z) \quad \begin{aligned} &= 0 \qquad z < 0 \\ &= V_0 \qquad z > 0 \end{aligned}$$

$$\left[i\gamma^\mu \partial_\mu - m - V(\vec{x}) \right] \psi(\vec{x}, t) = 0$$

We now try to calculate reflection and transmission coefficients.

If we calculate it, we will find that

$$R = \frac{(1-r)^2}{(1+r)^2} \qquad \tilde{k}^2 = (E-V_0)^2 - m^2$$

and

$$T = \frac{4r}{(1+r)^2} \qquad \text{with } r = \frac{\tilde{k}\,(E+m)}{(E-V_0+m)}$$

So for $V_0 > E + m$, $r < 0$

Hence $R > 1$ and $T < 0$!

Also $\tilde{k} = \sqrt{(E - V_0 - m)(E - V_0 + m)}$

Hence , if $|E - V_0| < m$, then $\tilde{k}$ is imaginary and we get an exponentially decaying sol^n as expected. But if $|E - V_0| > m$ ($V_0 > E + m$), then $\tilde{k}$ is real and we get transmitted oscillatory sol^n in region II

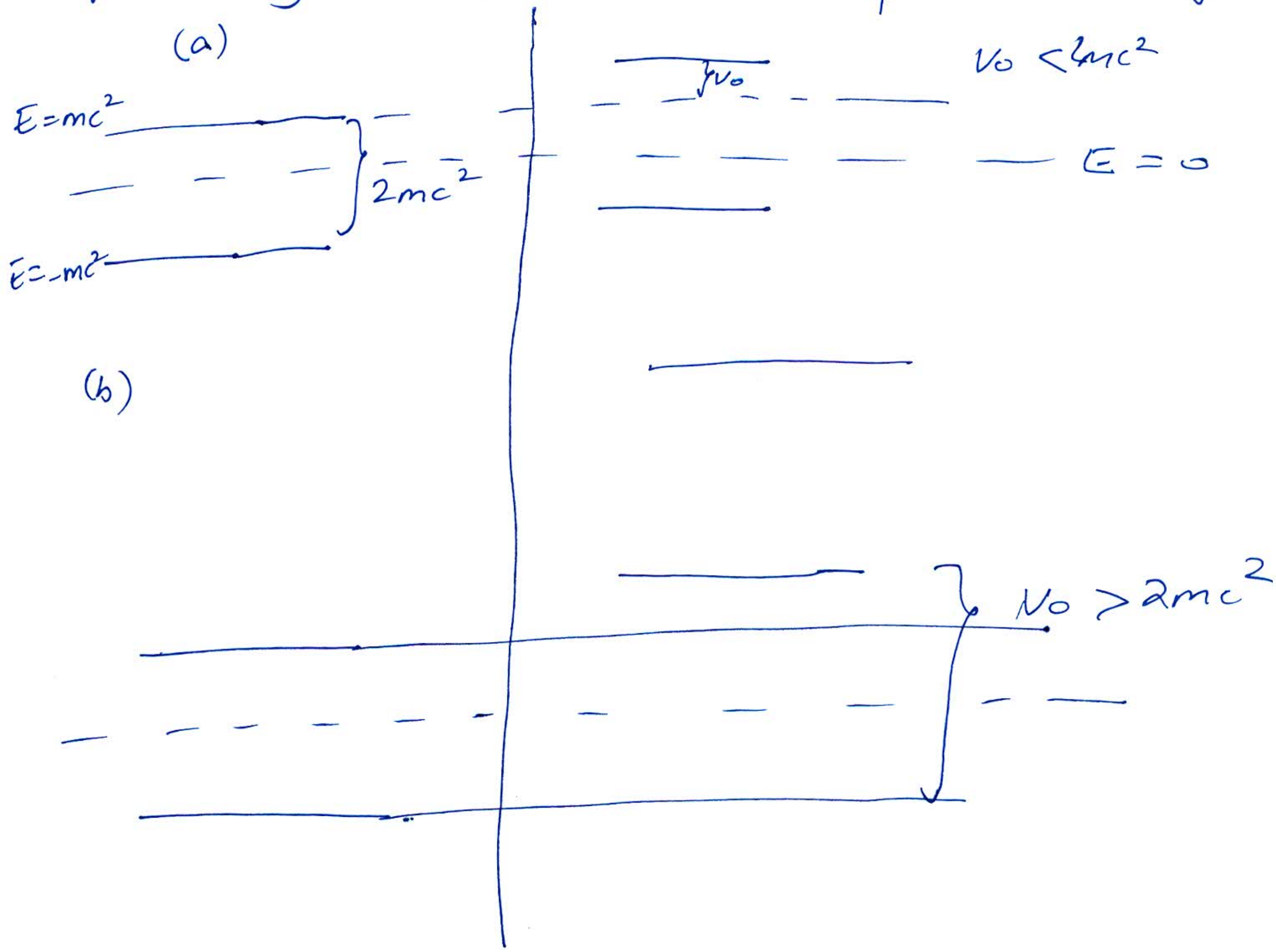
Hence, for $V_0 > E + m$, we get something unexpected. The smallest possible value for $E = m$, hence one can say that for $V_0 > 2m$ ($= 2mc^2$) , one gets something unexpected. This is the Klein paradox.

Essentially one understands this paradox in the context of quantum field theory by saying that potential of strength $V_0 > 2mc^2$ lead to pair production of an electron-hole pair.
(no longer really a static barrier)

~~Another way to think about it~~

The -ve transmission shows the flux of anti-ptcles to the right.

If we think of a finite barrier, then what we will find is that the barrier transmission, instead of being exponentially suppressed, is of order unity



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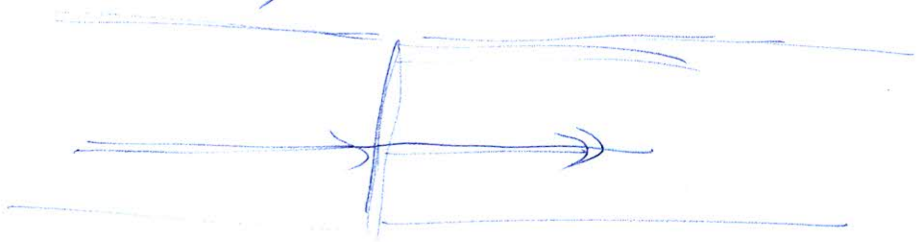
The main ~~gra~~ difference for graphene is that the Dirac fermions in 2+1 dimensions are massless. So ~~if~~ it turns out that $T = 1$ and $R = 0$ for any parameters of the potential, which of course you can see by solving for the barrier potent[ial]. There are also many easy ways to see why this is true.

The eq^{ns} to solve are

$$(i \vec{\sigma} \cdot \vec{\nabla} + V)\psi = E\psi$$

which gives (in the 1D case, which is what is relevant even in graphene if you consider normal incidence)



as opposed

to oblique incidence

$$i\frac{\partial \psi_2}{\partial x} = (E - V(x))\,\psi_1$$

and
$$i\frac{\partial \psi_1}{\partial x} = [E - V(x)]\,\psi_2$$

Let us introduce $\omega = \int_0 dx' |E - V(x')|$ (separately for $E > V(x)$ and $E < V(x)$ as we usually do for WKB)

Then we have

$$\psi_> = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{i|\omega|}$$

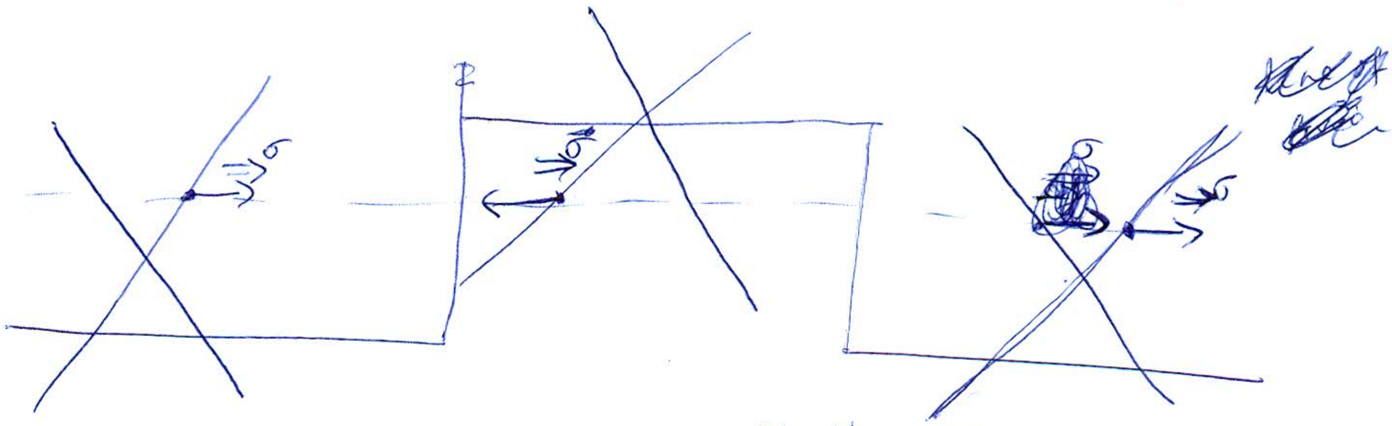
and
$$\psi_< = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-i|\omega|}$$

Since both components of the spinor has to be continuous at the turning points, the only way to match sol^{ns} is to have either $\psi_<$ or $\psi_>$ zero everywhere. So one can never have a combination of reflected and ~~transmitted waves~~ incident wave, since propagation is allowed only in one direction.

In other words, back-scattering is not allowed.

The point is the massless Dirac particle can propagate only along its

spin direction or opposite to it.
The scalar potential does not act on the spin; hence the spin of the Dirac particle cannot change. Hence its direction of motion cannot change. So conservation of helicity implies no back-scattering unless the spin changes.



(double arrows the spin

The arrows show the direction of mtm, but the group velocity is opposite to the mtm for holes.

So an electron changes to a hole under the potential barrier

This can be generalised to arbitrary angle of incidence.